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A SIMPLE MODEL OF ELASTIC-PLASTIC PLATES

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Abstract—The elastic-plastic plate is modelled by a sandwich of four layers and by two-dimensional quadrilateral elements. Piecewise constant approximations of the stresses within subregions provide a mechanism for the stepwise progression of yielding.

INTRODUCTION

In a previous article [1] the authors presented a simple means to approximate the constitutive equations for elastic–plastic plates and shells. The equations were drawn from a four-layer model of the plate. In the words of one reviewer[2], "this method seems to provide a practical means to establish constitutive equations of a general complicated elasto-plastic shell element. To verify it, it is necessary to show some concrete examples of its application to any structural problems". Such demonstration has awaited a correspondingly simple approximation in the remaining two-dimensions of the plate or shell. Recent articles[3, 4] set forth a finite-element, which is especially suited to the problems of elasto-plasticity. Here, we briefly recall the essential features of the elasto-plastic "club sandwich"[1], the bases and features of the two-dimensional element[4], and then show the union of these approximations to obtain acceptable numerical results with a modicum of computational effort.

FEATURES OF THE "CLUB SANDWICH"

The inherent difficulty in elasto–plasticity of plates and shells stems from bending, the gradients of stress through the thickness and the evolution of the plastic layers. Our model[1] of four discrete layers is an approximation which retains essential attributes of the one continuous layer, but offers computational simplicity and sufficient accuracy for most practical purposes. Figure 1 displays a plot of moment versus curvature during simple bending, loading and unloading. The club sandwich follows the piecewise linear plot; each corner corresponds to the transition from hookean to ideal plasticity in one layer, or vice-versa during unloading. Some curve fitting is possible: By a small violation of the initial yield condition in bending, the subsequent linear segment assumes a better overall fit. One plot ($\epsilon = 0.15$) in Fig. 1 incorporates 15% increase in the initial yield moment and exhibits better overall fit in the cycle of loading, unloading, and reloading.

FEATURES OF THE SIMPLE QUADRILATERAL ELEMENT

Considerable simplification of the quadrilateral element is possible by employing the functional of Hu-Washizu[5, 6], since each function (displacement, rotations, strains and stresses) can be independently approximated. The goal is an element which is simple, yet

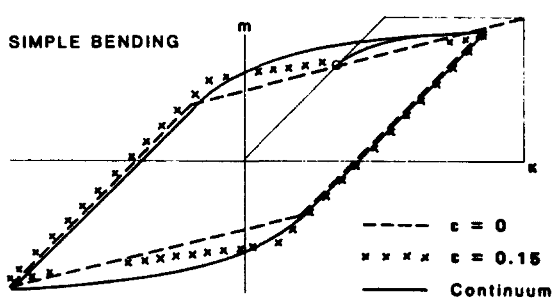

Fig. 1. Moment curvature.

exhibits neither instability nor excessive stiffness, such as shear locking. The elasto–plastic behavior also requires a simple and consistent enforcement of the yield and flow criteria; implementation calls for stepwise yielding in discrete subregions of the plate or shell. The formulation is described in the earlier article[3]. A simple way to accommodate the plasticity is introduced in the later article[4]; the essential feature is the piecewise *constant* approximation of stress (and strain) in the four quadrants of the quadrilateral. Here the stresses in question are the 2-D variables (forces and moments); the strains are the 2-D variables (strains and changes-of-curvature).

UNION OF FOUR-LAYER AND FOUR-QUADRANT CONCEPTS

The "club sandwich" provides the mechanism of step-by-step yielding through the four layers. The piecewise constant stresses (and strains) in the four quadrants of the 2-D element serve to subdivide completely each 3-D finite-element into 16 subelements of constant stress (and strain). *Each* subregion is therefore in a *homogeneous state*, and is governed by the appropriate elastic or elasto–plastic moduli. The constitutive equations, and computational program, for the sandwich govern each subregion (quadrant) of the 2-D element.

IMPLEMENTATION

At each stage of loading, each subrigion is governed by moduli, as determined by the theory of the club sandwich. As the load is increased, each subregion and, in fact, the entire plate is governed by linear equations until a subregion reaches the yield condition or reverts to elastic unloading. Each increment of load is selected to initiate the next transition in a subregion. The moduli are then recomputed in each yielded subregion and the procedure is repeated. If an increment is excessive and produces significant changes in moduli, then the usual corrective procedures are needed. In the following numerical examples, ideal plasticity was assumed. In most instances, the changes in stresses within yielded regions were small enough so that subsequent corrections were unnecessary.

SIMPLY SUPPORTED SQUARE PLATE UNDER UNIFORM LOAD

The simply supported plate is admittedly an easy test, but serves to illustrate the approximations, four discrete layers in thickness and four quadrants of each square element. Some plots of load versus deflection are displayed in Fig. 2. The uppermost curve is obtained with 20 Layers and provides a benchmark. To the onset of yielding, hookean behavior is adequately predicted with only 4 elements (2 × 2) in a quadrant of the plate. Following initial yielding the model exhibits softness, as anticipated from the moment-curvature plot of Fig. 1.

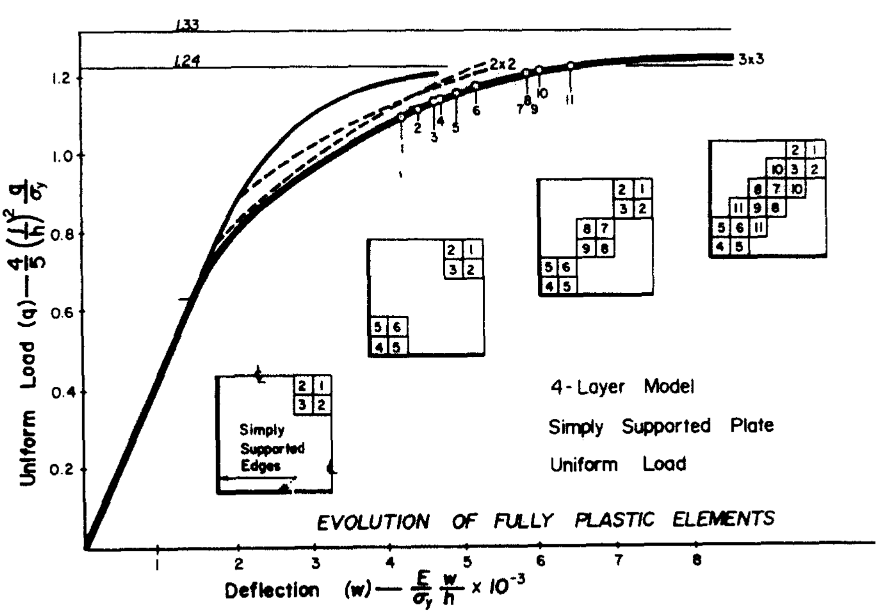

Fig. 2. Load-deflection of a simply supported plate.

Improvement is seen, if the yield condition is modified ($\epsilon = 0.15$). With 9 elements (3×3) the results are better; the curve crosses the lower bound of the limit load[7] when the curvature is about four times the yield value. The latter curve appears nearly horizontal at a load slightly above the lower bound (1.24) and well below the upper bound (1.33). Here the yielding progresses stepwise through the four layers in each of the 36 subregions (the 4 quadrants of the 9 elements). The evolution of fully plastic subregions is enumerated (1–11) and depicted in four stages; the last quadrant on the right shows the hinge along the diagonal.

CLAMPED SQUARE PLATE UNDER UNIFORM LOAD

The plate with clamped edges provides a better test and a challenge to the piecewise constant approximation of moments. Again, relatively, few elements (5×5 elements in the quadrant) provide a good discription of overall response and the load-deflection curve of Fig. 3, where the fully plastic subregions are enumerated and identified. Again, the lower bound (2.14) to the limit load is passed when the curvature is about four times the yield value, but reaches the upper bound (2.46) at about eight times the yield value. Undoubtedly, a finer subdivision would provide better accuracy, but our aim is to show practical results with relatively simple models. Also, at the deflections shown ($w = 6$) the membrane effects are more important than any refinements in our discription of bending.

It is interesting to examine the piecewise constant approximation of bending couples. The uppermost plot of Fig. 4 shows the approximation in a hookean plate through the elements adjacent to the center-line from the center B to a mid-point A on the edge. Though the quadrant contains 25 elements (5×5), each element contains 4 quadrants of constant stress. At best, each constant is the mean value in the subregion, and our constants are close to the means. Near the edge A, the steep gradient produces a significant difference between the values at the edge and the mean in the adjoining element. To explore the approximation, only the boundary elements were subdivided, once to make 49 elements (7×7) and again to make 81 elements (9×9), as depicted in Fig. 4. With each refinement the value in the boundary element C approaches the value A^*, which is predicted by the Kirchhoff theory as given by Timoshenko[8].

CONCLUSION

The basic difficulties in approximating the plasticity of plates (or shells) arise with the progression of yielding through the thickness and lead to expenses in computation and, especially, in storage. The club sandwich provides a two-dimensional theory of the elastic-plastic plate (or shell); the constitutive equations[1] apply to any path of loading in the space of forces and moments. As such, the sandwich might be implemented with a multitude of two-dimensional elements, arbitrary quadrilateral or triangular elements, or by alternative methods for the discrete approximation of plates and shells. Here, the theory of four layers is

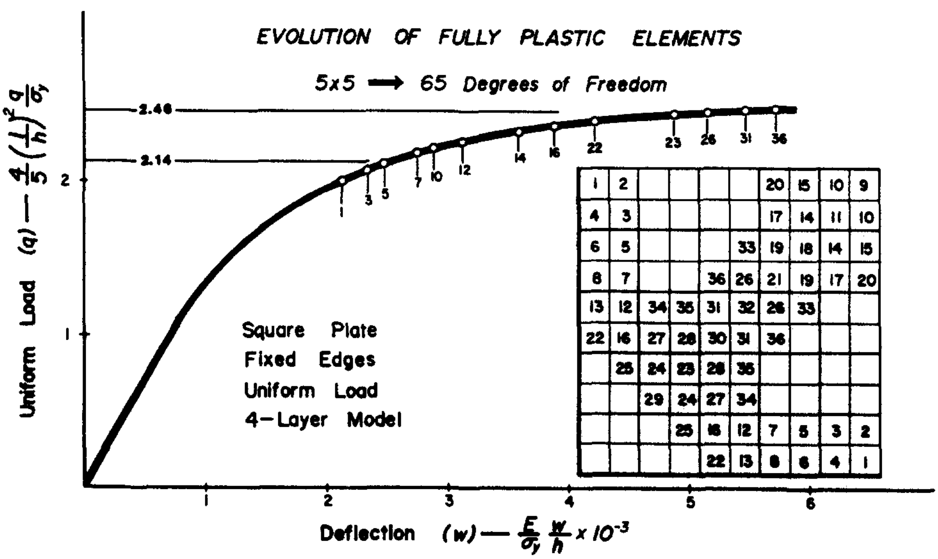

Fig. 3. Load-deflection of a clamped plate.

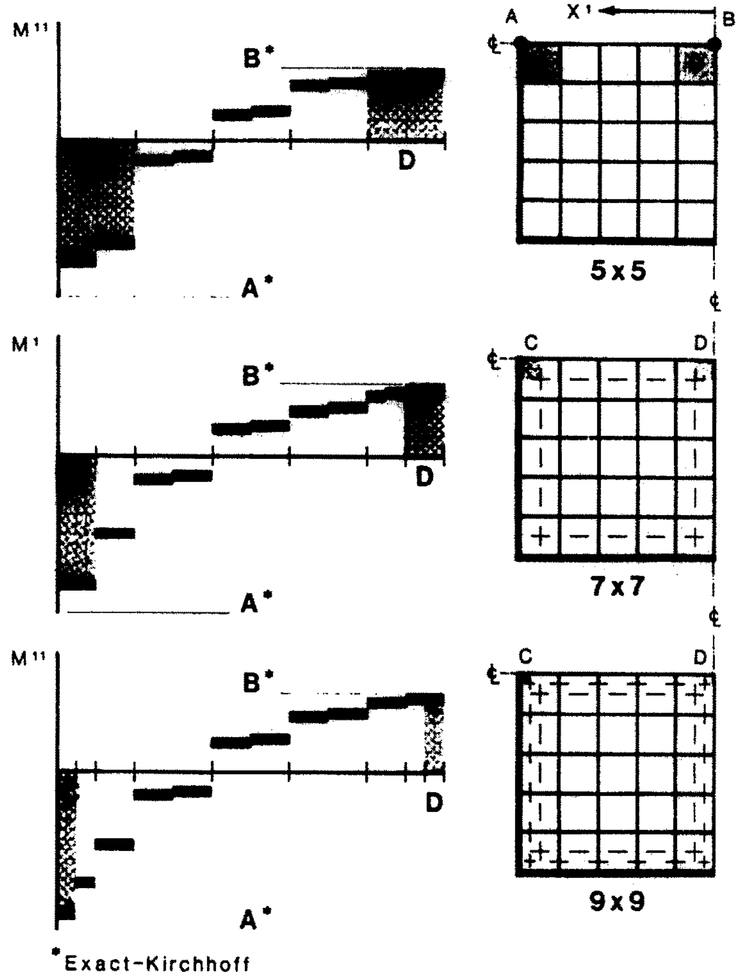

Fig. 4. Bending moment in a clamped plate.

used with an element[4] which embodies piecewise constant approximations of stress (forces and moments). That element is particularly suited to elastoplasticity, since it provides a consistent mechanism for a stepwise progression of yielding. The union of the club sandwich[1] and the simple element[4] produces a practical model of a plate or shell. Since bending is the inherent source of the difficulties, the square plate serves to demonstrate the most relevant attributes of that model. It remains to mate the club sandwich with other elements which might be more adaptable to arbitrary surfaces.

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REFERENCES

1. G. A. Wempner and C.-M. Hwang, Elasto-plasticity of the club sandwich. *Int. J. Solids Structures* **16**(2) 161–165 (1980).
2. M. Goton, *Appl. Mech. Rev.* 5142, **33**(6), 770 (1980).
3. G. Wempner, D. Talaslidis and C.-M. Hwang, A simple and efficient approximation of shells via finite quadrilateral elements. *J. Appl. Mech.* **49**, 115–120 (1982).
4. D. Talaslidis and G. Wempner, A simple finite element for elastic-plastic deformations of shells. *J. Comput. Meth. Appl. Mech. Engng* **34**, 1051–1064 (1982).
5. H. C. Hu, On some variational principles in the theory of elasticity and plasticity. *Scientia Sinica* **4** (1955).
6. K. Washizu, On the variational principles of elasticity and plasticity. *Tech. Rep.* 25-18, *Mass. Inst. Tech.* (1955).
7. P. G. Hodge, Jr. and T. Belytschko, Numerical methods for the limit analysis of plates. *J. Appl. Mech.* 35, 796–802 (1968).
8. S. Timoshenko and S. Woinosky-Krieger, *Theory of Plates and Shell*, p. 202. McGraw-Hill, New York (1959).